

Multiple positive solutions for Kirchhoff type problems involving concave and critical nonlinearities in \mathbb{R}^3

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Abstract: In this paper, we consider the multiplicity of solutions for a class of Kirchhoff type problems with sub-linear and critical terms on an unbounded domain. With the aid of Ekeland's variational principle and the concentration compactness principle we prove that the Kirchhoff problem has at least two solutions.

Keywords: Kirchhoff type problem, the concentration compactness principle, variational method.

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1 Introduction and main results

This paper concerns the multiplicity of solutions for the following Kirchhoff type problem

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u = \lambda f(x) |u|^{q-2} u + u^5, & x \in \mathbb{R}^3, \\ u \in D^{1,2}(\mathbb{R}^3), \end{cases} \quad (1.1)$$

where a, b are positive constants, $f(x)$ is a continuous function, $1 \leq q \leq 2$ or $1 \leq q < 2$.

It is well known that Kirchhoff type problems are proposed by Kirchhoff in 1883 [22] as an extension of the classical D'Alembert's wave equation for free vibration of elastic strings. Such problems are often viewed as nonlocal because the presence of the integral term $\int |\nabla u|^2 dx$. This phenomenon causes some mathematical difficulties making the study of such problems particularly interesting. The case of Kirchhoff problems where the nonlinear term is super-linear has been investigated in the last decades by many authors, for example [13, 14, 18, 19, 20, 21, 29, 26, 33, 40, 42] and references therein. Here, we are interested in the case of Kirchhoff problems where the nonlinear term is sub-linear and critical.

For nonlinear elliptic problems, Chabrowski and Drabek [8] considered the following nonlinear elliptic problem:

$$-\Delta u + V(x)u = \epsilon h(x)u^q + u^{2^*-1} \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

where $\epsilon > 0$ is a parameter, $1 < q < 2$ and $2^* = \frac{2N}{N-2}$, $N \geq 3$, is the critical Sobolev exponent. Under the assumptions of h is a nonnegative and nonzero function in $L^r(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, where $r = \frac{2^*}{2^*-q-1}$, they obtained that (1.2) has at least two nonnegative solutions by applying various variational principle.

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For Kirchhoff problems, Fan [17] investigated the existence of multiple positive solutions to the following Kirchhoff type problem:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = d(x)u^{k-1}u + g(x)u^5, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.3)$$

where $a, b > 0, 4 < k < 6$, Ω is a smooth bounded domain in \mathbb{R}^3 and $d(x), g(x)$ are positive and continuous functions. By introducing suitable conditions on $d(x), g(x)$, they proved that there exists Λ_{δ} such that if $|f|_{L^{q^*}} < \Lambda_{\delta}$, (1.3) has at least $cat_{M_{\delta}}(M)$ distinct positive solutions, where $q^* = \frac{6}{6-q}$ and cat mens the Ljusternik-Schnirelmann category (see [37]).

Liu et. al. [28] considered the following nonlinear Kirchhoff type equation

$$\begin{cases} -(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u = u^{2^*-2}u + \mu h(x), & x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.4)$$

where $a \geq 0, b > 0, N \geq 3, 2^* = \frac{2N}{N-2}, \mu \geq 0$ and $h \in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N \setminus \{0\})$ is nonnegative. Under some assumptions on a, b and μ , they obtained the existence of two positive solutions for Eq. (1.4).

Sun [33] studied the following Kirchhoff type problem with critical exponent

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda |u|^{q-2}u + u^5, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.5)$$

where $a, b, \lambda > 0, 1 < q < 2$, Ω is a smooth bounded domain in \mathbb{R}^3 . They showed that there exists a positive constant $T(a)$ depending on a such that for each $a > 0$ and $0 < \lambda < T(a)$, (1.5) has at least one positive solution.

Xie et. al. [41] considered the following Kirchhoff type problems

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = u^5, & x \in \mathbb{R}^3, \\ u \in D^{1,2}(\mathbb{R}^3), \end{cases} \quad (1.6)$$

where $a, b > 0$ and $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ is a given nonnegative function. If $|V|_{\frac{3}{2}}$ is suitable small, they proved that (1.6) has at least one bound state solution.

Motivated by above papers, we consider the Kirchhoff problem (1.1) with concave and critical nonlinearities on the whole space \mathbb{R}^3 . To the best of our knowledge, there are few papers which deal with this type of Kirchhoff problem (1.1). The main difficulty is how to estimate the energy and recover the compactness because the nonlinearity is the combination of the concave and critical terms. By the method of Mountain Pass Theorem and the concentration compactness principle, we obtain (1.1) has at least two different solutions with their energies having different signs.

Theorem 1.1 *Assume that in the problem (1.1), a, b are positive constants, $1 \leq q \leq 2$ and $f(x)$ is a function that can change sign and with property*

$$(f) \quad f(x) \in C(\mathbb{R}^3) \cap L^{q^*}(\mathbb{R}^3), \text{ where } q^* = \frac{6}{6-q}.$$

Then there exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, the problem (1.1) has one positive solutions which has a negative energy.

Theorem 1.2 Assume that in the problem (1.1), a, b are positive constants, $1 \leq q < 2$ and $f(x)$ is a nonnegative function with property f . Then there exists $0 < \lambda_2 \leq \lambda_1$ such that if $\lambda \in (0, \lambda_2)$, the problem (1.1) has two positive solutions, one of which has a positive energy and the other a negative energy.

Remark 1.3 Compare with Fan [17], we consider the Kirchhoff problem with the nonlinear term is sub-linear(or linear) and critical in the whole space \mathbb{R}^3 i.e. $1 \leq q < 2$ in Eq. (1.1), while he investigated the Kirchhoff problem with the nonlinear term is super-triple and critical in a smooth bounded domain in \mathbb{R}^3 i.e. $4 < k < 6$ in Eq. (1.3). Compare with Liu et. al.[28], if $q = 1$, $f(x)$ is a nonnegative function and $f(x) \in C(\mathbb{R}^3) \cap L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N \setminus \{0\})$ in Eq. (1.1), then

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u = \lambda f(x) + u^5, & x \in \mathbb{R}^3, \\ u \in D^{1,2}(\mathbb{R}^3), \end{cases}$$

which is the same as Eq. (1.4) when $N = 3$. Compare with Sun [33], from Theorem 1.2, we obtain the existence of two positive solutions for Eq. (1.1) in the whole space \mathbb{R}^3 , while he obtained the existence of one positive solution for Eq. (1.5) in a smooth bounded domain in \mathbb{R}^3 . Compare with Xie et. al. [41], if $\lambda = 1$, $q = 2$, $f(x)$ is a negative function and $f(x) \in C(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ in Eq. (1.1), then

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + f(x)u = u^5, & x \in \mathbb{R}^3, \\ u \in D^{1,2}(\mathbb{R}^3), \end{cases}$$

which is the same as (1.6). From Theorem 1.1, we can see that there exists $\sigma > 0$ such that if $|f|_{\frac{3}{2}} \in (0, \sigma)$, the problem (1.1) has one positive solutions which has a negative energy.

Throughout this paper, we make use of the following notations:

- \rightarrow (respectively, \rightharpoonup) denotes strong (respectively, weak) convergence;
- $|\cdot|_p$ denotes the norm of $L^p(\mathbb{R}^N)$;
- $D^{1,2}(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the norm $\|u\| = (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{\frac{1}{2}}$.
- X^* denotes the dual space of X ;
- $B_\alpha := \{u \in D^{1,2}(\mathbb{R}^3) : \|u\| = \alpha\}$ and $\overline{B}_\alpha := \{u \in D^{1,2}(\mathbb{R}^3) : \|u\| \leq \alpha\}$.
- C, C_1, C_2, \dots denote various positive constants, which may vary from line to line.

This paper is organized as follows: Section 2 is dedicated to the abstract framework and some preliminary results. Sections 3 and 4 are dedicate to the proofs of Theorems 1.1 and 1.2, respectively.

2 Preliminaries

The energy functional corresponding to (1.1) is defined on $D^{1,2}(\mathbb{R}^3)$ by

$$I(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{q} \lambda \int_{\mathbb{R}^3} f(x) |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx. \quad (2.1)$$

It is well known that a weak solution of problem (1.1) is a critical point of the functional I . In the following, we are devoted to finding critical points of I .

Lemma 2.1 ([37]) Suppose the hypothesis (f) holds, the function $\varphi : D^{1,2}(\mathbb{R}^3) \mapsto \mathbb{R}$ defined by

$$\varphi(u) = \int_{\mathbb{R}^3} f(x)|u|^q dx$$

is weakly continuous. Moreover, φ is continuously differentiable with derivative $\varphi' : D^{1,2}(\mathbb{R}^3) \mapsto (D^{1,2}(\mathbb{R}^3))^*$ given by

$$\langle \varphi'(u), v \rangle = q \int_{\mathbb{R}^3} f(x)|u|^{q-2} u v dx.$$

Obviously, the functional $I \in C^1(D^{1,2}(\mathbb{R}^3), \mathbb{R})$ and for any $u, v \in D^{1,2}(\mathbb{R}^3)$,

$$\langle I'(u), v \rangle = (a + b\|u\|^2) \int_{\mathbb{R}^3} \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^3} f(x)|u|^{q-2} u v dx - \int_{\mathbb{R}^3} |u|^4 u v dx.$$

Lemma 2.2 ([37]) The embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Denote by S the best Sobolev constant, which is given by

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx \right)^{\frac{1}{3}}}. \quad (2.2)$$

Moreover, S is achieved by the function

$$U_\epsilon(x) = \frac{(3\epsilon^2)^{\frac{1}{4}}}{(\epsilon^2 + |x|^2)^{\frac{1}{2}}}. \quad (2.3)$$

and

$$\int_{\mathbb{R}^3} |\nabla U_\epsilon(x)|^2 dx = \int_{\mathbb{R}^3} |U_\epsilon(x)|^6 dx = S^{\frac{3}{2}}. \quad (2.4)$$

3 Proof of Theorem 1.1

In this section, we are devoted to the proof of Theorem 1.1, so we suppose that the assumptions of Theorem 1.1 hold throughout this section.

Lemma 3.1 (i) There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, then there exist $\alpha > 0$ and $\rho > 0$ such that

$$I(u)|_{B_\alpha} \geq \rho > 0.$$

(ii) There is $u_0 \in \overline{B}_\alpha$ such that $I(u_0) < 0$.

proof(i) By the assumption (f), the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}|f|_{q^*}|u|_6^q - \frac{1}{6}|u|_6^6 \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{qS^{\frac{q}{2}}}|f|_{q^*}\|u\|^q - \frac{1}{6S^3}\|u\|^6 \\ &= \|u\|^q \left(\frac{b}{4}\|u\|^{4-q} - \frac{\lambda}{qS^{\frac{q}{2}}}|f|_{q^*} - \frac{1}{6S^3}\|u\|^{6-q} \right). \end{aligned} \quad (3.1)$$

Set $l(t) = \frac{b}{4}t^{4-q} - \frac{1}{6S^3}t^{6-q}$ for $t > 0$. Direct calculations yield that

$$\max_{t>0} l(t) = l(\alpha) = \left(\frac{3bS^3(4-q)}{2(6-q)} \right)^{\frac{4-q}{2}} \cdot \frac{b}{2(6-q)} := C_{p,q},$$

where $\alpha = (\frac{3bS^3(4-q)}{2(6-q)})^{\frac{1}{2}}$. Then it follows from (3.1) that, if $\lambda < \lambda_1$, $I(u)|_{B_\alpha} \geq \rho > 0$, where $\lambda_1 = qS_2^{\frac{q}{2}}C_{p,q} \cdot \frac{1}{|f|_{q^*}}$ and $\rho = \alpha^q(l(\alpha) - \frac{\lambda}{q}|f|_{q^*}S_2^{-\frac{q}{2}}) > 0$.

(ii) By choosing a function $\varphi \in D^{1,2}(\mathbb{R}^3)$, $\varphi \geq 0, \neq 0$ and with $\text{supp } \varphi \subset \{x : f(x) > 0\}$, then for $t > 0$ small enough, we have

$$I(t\varphi) \leq \frac{a}{2}t^2\|\varphi\|^2 + \frac{b}{4}t^4\|\varphi\|^4 - \frac{\lambda}{q}t^q \int_{\mathbb{R}^3} f(x)|\varphi|^q dx - \frac{t^6}{6}|\varphi|_6^6 < 0.$$

This completes the proof. \square

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 It follows from Lemma 3.1 that

$$c_1 = \inf_{u \in \overline{B}_\alpha} I(u) < 0.$$

By Ekeland's variational principle [37], there exists a minimizing sequence $\{u_n\} \subset \overline{B}_\alpha$ such that $I(u_n) \rightarrow c_1$ and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then, there exists $u_1 \in D^{1,2}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u_1$ as $n \rightarrow \infty$ in $D^{1,2}(\mathbb{R}^3)$. By the fact that \overline{B}_α is closed and convex, thus $u_1 \in \overline{B}_\alpha$, then $c_1 \leq I(u)$. It follows from the Hölder inequality that

$$\begin{aligned} c_1 &\leq I(u_1) \\ &= I(u_1) - \frac{1}{4} \liminf_{n \rightarrow \infty} (I'(u_n), u_1) \\ &= \frac{a}{4}\|u_1\|^2 + \frac{b}{4}\|u_1\|^2(\|u_1\|^2 - \liminf_{n \rightarrow \infty} \|u_n\|^2) + (\frac{1}{4} - \frac{1}{q})\lambda \int_{\mathbb{R}^3} f(x)|u_1|^q dx + \frac{1}{12}|u_1|_6^6 \\ &\leq \liminf_{n \rightarrow \infty} (\frac{a}{4}\|u_n\|^2 + (\frac{1}{4} - \frac{1}{q})\lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx + \frac{1}{12}|u_n|_6^6) \\ &= \liminf_{n \rightarrow \infty} (I(u_n) - \frac{1}{4}(I'(u_n), u_n)) = c_1. \end{aligned} \tag{3.2}$$

From above we can deduce that $u_n \rightarrow u_1$ in $D^{1,2}(\mathbb{R}^3)$. Thus the functional I satisfies the $(PS)_{c_1}$ condition and the functional I achieves a minimum u_1 at an interior point of B_α . Since $I(u_1) = I(|u_1|)$ we may assume that $u_1 \geq 0$ and by the maximum principle we have $u_1 > 0$ on \mathbb{R}^3 . This completes the proof. \square

4 Proof of Theorem 1.2

First, we prove the following mountain-pass geometry of functional I .

Lemma 4.1 (*Mountain pass Geometry*) *The functional I satisfies the following conditions:*

(i) *There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$I(u)|_{B_\alpha} \geq \rho > 0.$$

(ii) *There exists $e \in D^{1,2}(\mathbb{R}^3)$ with $\|e\| > \alpha$ such that $I(e) < 0$.*

proof(i) It directly follows from Lemma 2.1.

(ii) Note that

$$I(tU_\epsilon) = \frac{a}{2}t^2\|U_\epsilon\|^2 + \frac{b}{4}t^4\|U_\epsilon\|^4 - \frac{t^q}{q}\lambda \int_{\mathbb{R}^3} f(x)|U_\epsilon|^q dx - \frac{t^6}{6}|U_\epsilon|_6^6.$$

Then, there exists $t_0 > 0$ sufficient large such that $\|tU_\epsilon\| > \rho$, $I(tU_\epsilon) < 0$, where U_ϵ defined in (2.3). \square

Therefore, by using the Ambrosetti-Rabinowitz Mountain Pass Theorem without $(PS)_{c_2}$ condition(see [37]), it follows that there exists a $(PS)_{c_2}$ sequence $\{u_n\} \subset D^{1,2}(\mathbb{R}^3)$ such that

$$I(u_n) \rightarrow c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) \quad \text{and} \quad I'(u_n) \rightarrow 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], D^{1,2}(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Lemma 4.2 *The $(PS)_{c_2}$ sequence $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$.*

proof By the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} & c_2 + 1 + \|u_n\| \\ & \geq I(u_n) - \frac{1}{6}(I'(u_n), u_n) \\ & = \frac{a}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \left(\frac{1}{q} - \frac{1}{6}\right)\lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\ & \geq \frac{a}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \left(\frac{1}{q} - \frac{1}{6}\right)\lambda |f|_{q^*}|u_n|_6^q \\ & \geq \frac{a}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \left(\frac{1}{q} - \frac{1}{6}\right)\lambda \frac{1}{S^{\frac{q}{2}}} |f|_{q^*} \|u_n\|^q. \end{aligned}$$

Then $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$. \square

Lemma 4.3 *If $\{u_n\} \subset D^{1,2}(\mathbb{R}^3)$ is a bounded $(PS)_{c_2}$ sequence of I and $c_2 < \Lambda - C\lambda^{\frac{2}{2-q}}$, then $\{u_n\}$ has a strongly convergent subsequence in $D^{1,2}(\mathbb{R}^3)$, where $\Lambda = \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{\frac{3}{2}}$ and $C = \frac{2-q}{2}(\frac{1}{q} - \frac{1}{6})^{\frac{2}{2-q}} |f|_{q^*}^{\frac{2}{2-q}} (\frac{2q}{aS})^{\frac{q}{2-q}}$.*

proof By the concentration compactness lemma by P.L. Lions [37], up to a subsequence, there exist an at most countable set Γ , points $\{a_k\}_{k \in \Gamma} \subset \mathbb{R}^3$ and values $\{\eta_k\}_{k \in \Gamma}, \{\nu_k\}_{k \in \Gamma} \in \mathbb{R}^+$ such that

$$\begin{cases} |\nabla u_n|^2 \rightharpoonup d\eta \geq |\nabla u|^2 + \sum_{k \in \Gamma} \eta_k \delta_{a_k}, \\ |u_n|^6 \rightharpoonup d\nu = |u|^{2^*} + \sum_{k \in \Gamma} \nu_k \delta_{a_k}, \end{cases} \quad (4.1)$$

where δ_{a_k} is the Dirac delta measure concentrated a_k . Moreover,

$$\nu_k \leq \eta_k^3 S^{-3}. \quad (4.2)$$

In the following, we prove that $\Gamma = \emptyset$. Arguing by contradiction, fix $k \in \Gamma$, for $\epsilon > 0$, assume that $\psi_\epsilon^k \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that

$$\begin{cases} \psi_\epsilon^k = 1, & \text{for } |x - a_k| \leq \frac{\epsilon}{2}, \\ \psi_\epsilon^k = 0, & \text{for } |x - a_k| \geq \epsilon, \\ |\nabla \psi_\epsilon^k| \leq \frac{3}{\epsilon}, & \text{in } \mathbb{R}^3. \end{cases}$$

Since $\{\psi_\epsilon^k u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$, we have

$$(I'(u_n), \psi_\epsilon^k u_n) \rightarrow 0,$$

i.e.

$$(a + b\|u_n\|^2) \left(\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_\epsilon^k dx + \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_\epsilon^k dx \right) = \lambda \int_{\mathbb{R}^3} f(x) |u_n|^q \psi_\epsilon^k dx + \int_{\mathbb{R}^3} |u_n|^6 \psi_\epsilon^k dx + o(1). \quad (4.3)$$

It follows from the boundedness of $\{u_n\}$ in $D^{1,2}(\mathbb{R}^3)$ and the Hölder inequality that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (a + b\|u_n\|^2) \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_\epsilon^k dx \\ & \leq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} C_1 \left(\int_{B_\epsilon(a_k)} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_\epsilon(a_k)} |\nabla \psi_\epsilon^k|^2 |u_n|^2 dx \right)^{\frac{1}{2}} \\ & \leq \lim_{\epsilon \rightarrow 0} C_2 \left(\int_{B_\epsilon(a_k)} |\nabla \psi_\epsilon^k|^2 |u|^2 dx \right)^{\frac{1}{2}} \\ & \leq \lim_{\epsilon \rightarrow 0} C_3 \left(\int_{B_\epsilon(a_k)} |\nabla \psi_\epsilon^k|^3 dx \right)^{\frac{1}{3}} \left(\int_{B_\epsilon(a_k)} |u|^6 dx \right)^{\frac{1}{6}} \\ & \leq \lim_{\epsilon \rightarrow 0} C_4 \left(\int_{B_\epsilon(a_k)} |u|^6 dx \right)^{\frac{1}{6}} \\ & = 0, \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} f(x) |u_n|^q \psi_\epsilon^k dx \\ & = \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon(a_k)} f(x) |u|^q \psi_\epsilon^k dx \\ & = 0. \end{aligned} \quad (4.5)$$

From (??), we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (a + b\|u_n\|^2) \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_\epsilon^k dx \\ & \geq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} a \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_\epsilon^k dx + \limsup_{n \rightarrow \infty} b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_\epsilon^k dx \right)^2 \\ & \geq a\eta_k + b\eta_k^2, \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 \psi_\epsilon^k dx \\ & = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} |u|^6 \psi_\epsilon^k dx + \nu_k \\ & = \nu_k. \end{aligned} \quad (4.7)$$

From (4.3)-(4.7), we have

$$\nu_k \geq a\eta_k + b\eta_k^2. \quad (4.8)$$

Combining with (4.2), we can deduce that

$$\eta_k \geq \frac{b + \sqrt{b^2 + 4aS^{-3}}}{2S^{-3}}. \quad (4.9)$$

For $R > 0$, assume that $\phi_R \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that

$$\begin{cases} \phi_R = 1, & \text{for } |x| < R, \\ \phi_R = 0, & \text{for } |x| \geq 2R, \\ |\nabla \phi_R| \leq \frac{2}{R}, & \text{in } \mathbb{R}^3. \end{cases}$$

Then, by Lemma 2.2, we have

$$\begin{aligned} c_2 &= \lim_{n \rightarrow \infty} (I(u_n) - \frac{1}{4}(I'(u_n), u_n)) \\ &= \lim_{n \rightarrow \infty} (\frac{a}{4}\|u_n\|^2 + \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 dx - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\ &\geq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} (\frac{a}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 \phi_R dx + \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 \phi_R dx - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\ &\geq \lim_{R \rightarrow \infty} (\frac{a}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \phi_R dx + \frac{a}{4}\eta_k + \frac{1}{12} \int_{\mathbb{R}^3} |u|^6 \phi_R dx + \frac{1}{12}\nu_k - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\ &\geq \frac{a}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{a}{4}\eta_k + \frac{1}{12} \int_{\mathbb{R}^3} |u|^6 \phi_R dx + \frac{1}{12}\nu_k - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\ &\geq \frac{aS}{4}|u|_6^2 + \frac{a}{4}\eta_k + \frac{1}{12}\nu_k - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx. \end{aligned} \quad (4.10)$$

By (4.8) and (4.9), we have

$$\frac{a}{4}\eta_k + \frac{1}{12}\nu_k \geq \Lambda. \quad (4.11)$$

In order to estimate $\frac{aS}{4}|u|_6^2 - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx$, we observe that the function

$$\begin{aligned} t &\mapsto \frac{aS}{4}t^2 - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\ &\geq \frac{aS}{4}t^2 - (\frac{1}{q} - \frac{1}{4})\lambda |f|_{q^*} t^q := \tilde{f}(t) \end{aligned} \quad (4.12)$$

achieves its minimum on $(0, \infty)$ at a point t_1 , $\min_{t \geq 0} \tilde{f}(t) = \tilde{f}(t_1) = -C\lambda^{\frac{2}{2-q}}$, where

$$t_1 = \frac{2q}{aS}(\frac{1}{q} - \frac{1}{4})|f|_{q^*}^{\frac{1}{2-q}} \quad \text{and} \quad C = \frac{2-q}{2}(\frac{1}{q} - \frac{1}{6})^{\frac{2}{2-q}}|f|_{q^*}^{\frac{2}{2-q}}(\frac{2q}{aS})^{\frac{q}{2-q}}.$$

From (4.10), (4.11) and (4.12), we can deduce that $c_2 \geq \Lambda - C\lambda^{\frac{2}{2-q}}$, which is a contradiction. Thus $\Gamma = \emptyset$.

For $R > 0$, define

$$\eta_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^2 dx \quad (4.13)$$

and

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^6 dx. \quad (4.14)$$

Then,

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} d\eta + \eta_\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty. \quad (4.15)$$

Moreover,

$$\nu_\infty \leq \eta_\infty^3 S^{-3}. \quad (4.16)$$

Assume that $\chi_R \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that

$$\begin{cases} \chi_R(x) = 0, & \text{for } |x| < \frac{R}{2}, \\ \chi_R(x) = 1, & \text{for } |x| > R, \\ |\nabla \chi_R(x)| < \frac{3}{R}, & \text{in } \mathbb{R}^3. \end{cases}$$

Since $\{\chi_R u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$, we have

$$(I'(u_n), \chi_R u_n) \rightarrow 0,$$

i.e.

$$(a + b\|u_n\|^2) \left(\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \chi_R dx + \int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_R dx \right) = \lambda \int_{\mathbb{R}^3} f(x) |u_n|^q \chi_R dx + \int_{\mathbb{R}^3} |u_n|^6 \chi_R dx + o(1). \quad (4.17)$$

It follows from the boundedness of $\{u_n\}$ in $D^{1,2}(\mathbb{R}^3)$ and the Hölder inequality that

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} (a + b\|u_n\|^2) \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \chi_R dx \\ & \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\frac{R}{2} \leq |x| \leq R} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\frac{R}{2} \leq |x| \leq R} |\nabla \chi_R|^2 |u_n|^2 dx \right)^{\frac{1}{2}} \\ & \leq \lim_{R \rightarrow \infty} C_2 \left(\int_{\frac{R}{2} \leq |x| \leq R} |\nabla \chi_R^k|^2 |u|^2 dx \right)^{\frac{1}{2}} \\ & \leq \lim_{R \rightarrow \infty} C_3 \left(\int_{\frac{R}{2} \leq |x| \leq R} |\nabla \chi_R^k|^3 dx \right)^{\frac{1}{3}} \left(\int_{\frac{R}{2} \leq |x| \leq R} |u|^6 dx \right)^{\frac{1}{6}} \\ & \leq \lim_{R \rightarrow \infty} C_4 \left(\int_{\frac{R}{2} \leq |x| \leq R} |u|^6 dx \right)^{\frac{1}{6}} \\ & = 0, \end{aligned} \quad (4.18)$$

and

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} f(x) |u_n|^q \chi_R dx \\ & = \lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} f(x) |u|^q \chi_R dx \\ & = 0. \end{aligned} \quad (4.19)$$

From (4.13) and (4.14), we have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} (a + b\|u_n\|^2) \int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_R dx \\ & \geq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} a \int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_R dx + \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_R dx \right)^2 \\ & \geq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} a \int_{|x| > R} |\nabla u_n|^2 dx + \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} b \left(\int_{|x| > R} |\nabla u_n|^2 dx \right)^2 \\ & = a\eta_\infty + b\eta_\infty^2 \end{aligned} \quad (4.20)$$

and

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 \chi_R dx \\
&= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > \frac{R}{2}} |u_n|^6 \chi_R dx \\
&\leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > \frac{R}{2}} |u_n|^6 dx \\
&= \nu_\infty.
\end{aligned} \tag{4.21}$$

From (4.17)-(4.21), we have

$$\nu_\infty \geq a\eta_\infty + b\eta_\infty^2. \tag{4.22}$$

From $\eta_\infty, \nu_\infty \geq 0$, (4.16) and (4.22), we can see that $\eta_\infty = 0$ if and only if $\nu_\infty = 0$. In the following, we assume that $\eta_\infty \neq 0$. From (4.16) and (4.22), we can deduce that

$$\eta_\infty \geq \frac{b + \sqrt{b^2 + 4aS^{-3}}}{2S^{-3}}. \tag{4.23}$$

Then, it follows from (4.12)-(4.14), (4.22) and (4.23), we have

$$\begin{aligned}
c_2 &= \lim_{n \rightarrow \infty} (I(u_n) - \frac{1}{4}(I'(u_n), u_n)) \\
&= \lim_{n \rightarrow \infty} (\frac{a}{4}\|u_n\|^2 + \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 dx - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\
&\geq \frac{a}{4} \int_{\mathbb{R}^3} d\eta + \frac{a}{4}\eta_\infty + \frac{1}{12} \int_{\mathbb{R}^3} d\nu + \frac{1}{12}\nu_\infty - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\
&\geq \frac{a}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{a}{4}\eta_\infty + \frac{1}{12}\nu_\infty - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\
&\geq \frac{aS}{4}|u|_6^2 + \frac{a}{4}\eta_\infty + \frac{1}{12}\nu_\infty - (\frac{1}{q} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\
&\geq \Lambda - C\lambda^{\frac{2}{2-q}},
\end{aligned} \tag{4.24}$$

which contradict with the assumption of $c_2 \leq \Lambda - C\lambda^{\frac{2}{2-q}}$. Thus $\eta_\infty = \nu_\infty = 0$. From $\Gamma = \emptyset$ and (4.14), we can obtain that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} |u|^6 dx.$$

Combining with Fatou Lemma, we have

$$\int_{\mathbb{R}^3} |u|^6 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx.$$

Thus, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} |u|^6 dx$.

In the following that we prove that $u_n \rightarrow u$ in $D^{1,2}(\mathbb{R}^3)$. Assume that $\lim_{n \rightarrow \infty} \|u_n\|^2 = d^2$, it sufficient to show that $d^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. Indeed,

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} (I'(u_n), u_n - u) = \lim_{n \rightarrow \infty} I'(u_n)u_n - \lim_{n \rightarrow \infty} I'(u_n)u \\
&= \lim_{n \rightarrow \infty} (a + b\|u_n\|^2)\|u_n\|^2 - \int_{\mathbb{R}^3} |u_n|^6 dx - \lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\
&\quad - \lim_{n \rightarrow \infty} [(a + b\|u_n\|^2) \int_{\mathbb{R}^3} \nabla u_n \nabla u dx - \int_{\mathbb{R}^3} |u_n|^4 u_n u dx - \lambda \int_{\mathbb{R}^3} f(x)|u_n|^{q-2} u_n u dx] \\
&= (a + bd^2)(d^2 - \int_{\mathbb{R}^3} |\nabla u|^2 dx).
\end{aligned}$$

Thus, $u_n \rightarrow u$ in $D^{1,2}(\mathbb{R}^3)$. This shows that the functional I satisfies the $(PS)_{c_2}$ condition for $c_2 < \Lambda - C\lambda^{\frac{2}{2-q}}$. \square

In the following, we estimate the energy c_2 .

Lemma 4.4 *There exists $\lambda_2 \in (0, \lambda_1]$ such that for any $\lambda \in (0, \lambda_2)$, we have*

$$c_2 \leq \sup_{t \geq 0} I(tU_\epsilon) < \Lambda - C\lambda^{\frac{2}{2-q}}.$$

proof It follows from (2.4) that

$$\begin{aligned} I(tU_\epsilon) &= \frac{at^2}{2} \|U_\epsilon\|^2 + \frac{bt^4}{4} \|U_\epsilon\|^4 - \frac{t^6}{6} |U_\epsilon|_6^6 - \frac{t^q}{q} \lambda \int_{\mathbb{R}^3} f(x) |U_\epsilon|^q dx \\ &= \frac{at^2}{2} S^{\frac{3}{2}} + \frac{bt^4}{4} S^3 - \frac{t^6}{6} S^{\frac{3}{2}} - \frac{t^q}{q} \lambda \int_{\mathbb{R}^3} f(x) |U_\epsilon|^q dx. \end{aligned}$$

We observe that the function

$$\begin{aligned} t &\mapsto \frac{at^2}{2} \|U_\epsilon\|^2 + \frac{bt^4}{4} \|U_\epsilon\|^4 - \frac{t^6}{6} |U_\epsilon|_6^6 \\ &= \frac{at^2}{2} S^{\frac{3}{2}} + \frac{bt^4}{4} S^3 - \frac{t^6}{6} S^{\frac{3}{2}} := \tilde{g}(t) \end{aligned}$$

achieves its maximum on $[0, \infty)$ at a point t_2 , that is, $\max_{t \geq 0} \tilde{g}(t) = \tilde{g}(t_2) = \Lambda$, where $t_2^2 = \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2S^{\frac{3}{2}}}$. First, we choose $t_3 \in (0, t_2)$ and $\lambda_3 > 0$ small enough such that

$$I(tU_\epsilon) < \Lambda - C\lambda^{\frac{2}{2-q}}$$

for $0 \leq t \leq t_3$ and $0 < \lambda \leq \lambda_3$. Here λ_3 is chosen so that

$$\Lambda - C\lambda^{\frac{2}{2-q}} > 0$$

for all $0 < \lambda \leq \lambda_3$. To estimate $I(tU_\epsilon)$ for $t \geq t_3$, by $1 \leq q < 2$, we can choose $\lambda_2 \in (0, \lambda_3]$ such that for any $\lambda \in (0, \lambda_2]$, we have

$$\frac{t_3^q}{q} \lambda \int_{\mathbb{R}^3} f(x) |U_\epsilon|^q dx > C\lambda^{\frac{2}{2-q}}.$$

Then, for any $\lambda \in (0, \lambda_2)$, we have

$$\begin{aligned} \sup_{t \geq t_3} I(tU_\epsilon) &\leq \sup_{t \geq t_3} (\tilde{g}(t) - \frac{t^q}{q} \lambda \int_{\mathbb{R}^3} f(x) |U_\epsilon|^q dx) \\ &< \tilde{g}(t_1) - C\lambda^{\frac{2}{2-q}} \\ &= \Lambda - C\lambda^{\frac{2}{2-q}}. \end{aligned}$$

Thus, we complete the proof. \square

Proof of Theorem 1.2 The argument of Theorem 1.1 shows that there exist $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$ problem (1.1) has a solution which is a local minimum of I . From Lemmas 4.2, 4.3 and 4.4, there exist $\lambda_2 \leq \lambda_1$ such that for each $\lambda \in (0, \lambda_2)$, problem (1.1) has a solution which is a mountain pass solution. \square

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